Asymptotics of the largest zeros of some orthogonal polynomials

1998 J. Phys. A: Math. Gen. 31 5525

(http://iopscience.iop.org/0305-4470/31/25/005)

J. Phys. A: Math. Gen. **31** (1998) 5525–5544. Printed in the UK PII: S0305-4470(98)87132-1

Asymptotics of the largest zeros of some orthogonal polynomials*

Yang Chen† and Mourad E H Ismail‡

Department of Mathematics, Imperial College, 180 Queen's Gate, London SW7 2BZ, UK

Received 27 August 1997

Abstract. We study the asymptotics of the largest zeros of the Wilson, q^{-1}-Hermite and q-Laguerre polynomials using two distinct techniques. The first is based on the Coulomb fluid technique developed in a previous paper where the primary input is the weight function, while the second uses the method of chain sequences which supplies inequalities for the largest zeros; using the recurrence coefficients. We also investigate the asymptotics of the largest zeros of the polynomials orthogonal to the weight functions $\exp[-c\ (\log x)^m]$ for $c > 0$ and m a positive even integer.

1. Introduction

The theory of random matrices which originally arose from statistical modelling of the energy levels of heavy nuclei, has recently seen application in other diverse areas of physics such as quantum chaos [2], transport in disordered disordered solids [8] and low-dimensional string theory [3]. In pure mathematics, the Gaussian unitary ensemble, a special case of Hermitean random matrices, is important in the study of the zeros of the Riemann zeta-function [26].

In the theory of random matrices a central object of interest, denoted as $E[J]$, is the probability that an interval, J (a subset of $\mathbb{R}$) is free of eigenvalues. For complex Hermitean matrices, this quantity can be expressed as the Fredholm determinant of a certain integral operator over the interval J [23]:

$$E[J] = \det(I - \hat{K}_J)$$

where $\hat{K}$ has the kernel

$$K(x, y) = \sqrt{w(x)w(y)} \sum_{j=0}^{N-1} p_j(x) p_j(y).$$

Here $\{p_j(x) : 0 \leqslant j \leqslant N\}$ is the family of polynomials orthonormal with respect to the weight function $w(x)$:

$$\int w(x)\ p_j(x) p_k(x)\ \mathrm{d}x = \delta_{j,k}$$

* Research partially supported by NSF grant DMS 9625459 and a visiting fellowship from the Leverhulme Foundation.

† E-mail: y.chen@ic.ac.uk

‡ Permanent address: Department of Mathematics, University of South Florida, Tampa, FL 33620-5700, USA. E-mail: ismail@hahn.math.usf.edu

where the integral is over the support of $w(x)$, and $w(x)$ is related to the potential, $u(x)$, of the random matrix problem in the eigenvalue representation, through the relationship, $w(x) = \exp[-u(x)]$.

For $J = (s, \infty)$, $E[s, \infty]$ gives the probability distribution of the largest eigenvalue, s, under an appropriate scaling. With this in mind we are led to the ongoing program of establishing the 'edge' asymptotic behaviour of a large class of orthogonal polynomials. In order to compute such asymptotics, rather precise knowledge is required on the largest zeros of the associated orthogonal polynomials. This will be explained later. We shall use $x \approx y$ to mean y is an approximation to x while $f(x) \sim g(x)$ as $x \to a$ to mean $f(x)/g(x) \to 1$ as $x \to a$.

It is known from the important work of Ullman, Saff, Mhaskar, Rakhmanov, Lubinsky, Totik, Van Assche, Levin and others that the distribution function of the zeros denotes as $\sigma(x)$ can be obtained from the following minimization problem:

$$\min_{\sigma} F[\sigma] \qquad \text{subject to} \quad \int_J \sigma(x)\, \mathrm{d}x = N \tag{1.1}$$

where

$$F[\sigma] = \int_J u(x)\, \sigma(x)\, \mathrm{d}x - \int_J \int_J \sigma(x) \ln|x-y| \sigma(y)\, \mathrm{d}y\, \mathrm{d}x. \tag{1.2}$$

Here $\exp[-u(x)] = w(x)$ is the weight function and N is the degree of polynomials orthonormal with respect to weight $w(x)$:

$$\int_K p_M(x)\, p_N(x)\, w(x)\, \mathrm{d}x = \delta_{M,N} \tag{1.3}$$

where K is the interval of orthogonality. In this paper we shall focus our attention on cases for which K is the real line and positive real line. Note that $\sigma(\cdot)$, the zero counting function, is positive over its support J. The minimizing function σ satisfies a singular integral equation

$$u'(x) = P \int_{e_L}^{e_R} \frac{\sigma(y)}{x-y}\, \mathrm{d}y \tag{1.4}$$

where the interval J is (e_L, e_R). In the examples to be given below J is either $(0, b)$ or $(-b, b)$ and P denotes the principal value. The edge parameter b is determined by the normalization condition $\int_J \sigma(x)\, \mathrm{d}x = N$. The solution of the integral equation reads

$$\sigma(x) = \frac{1}{2\pi^2} \sqrt{\frac{e_R - x}{x - e_L}}\, P \int_{e_L}^{e_R} \frac{u'(y)}{y-x} \sqrt{\frac{y-e_L}{e_R - y}}\, \mathrm{d}y \qquad x \in (e_L, e_R). \tag{1.5}$$

The function $\sigma(x)$ given by (1.5) is indeed the potential theoretic approximation of

$$\sigma_N(x) := w(x) \sum_{n=0}^{N-1} [p_n(x)]^2$$

expected to be valid for sufficiently large N. This technique was developed by Dyson [11] on certain random matrix ensembles in the 1960s and has recently found application in other matrix ensembles [4–8].

As an example take $u(x) = x^2$ and $K = (-\infty, \infty)$. In this case the orthogonal polynomials are the well known Hermite polynomials. A simple calculation using (1.5) with $J = (-b, b)$ gives

$$\sigma(x) = \frac{1}{\pi} \sqrt{b^2 - x^2} \qquad x \in (-b, b) \tag{1.6}$$

with $b = \sqrt{2N}$ from the normalization condition. Since $\sigma(\pm b) = 0$ in (1.6), the parameter b defines the edges beyond which the density vanishes. To a crude approximation b can be identified as the largest zero. Our contribution to this method is the observation that a further refinement to the largest zero can be obtained from the formula

$$1 = \int_a^b \sigma(x)\,\mathrm{d}x \tag{1.7}$$

in the limits $b \to \infty$ and $a \to b$. The a in (1.7) is a better approximation to the largest zero than b is. Thus equation (1.7) supplies the asymptotics of the largest zero. To see how this works, we first determine the behaviour of $\sigma(x)$ near $x = b$. This is

$$\sigma(x) \sim G(b)\sqrt{b-x} \qquad \text{as } \; x \to b^- \tag{1.8}$$

where in the Hermite case

$$G(b) := \frac{\sqrt{2b}}{\pi}.$$

We will show in the case of other sequences of polynomials, to be studied later, that the density has the above 'edge' behaviour.

A simple integration using (1.8) gives

$$a \sim b - \left(\frac{3}{2G(b)}\right)^{2/3}. \tag{1.9}$$

We remark that since $G(b) > 0$ the crude estimate b for the largest zero is larger than the refinement given in (1.9).

In the Hermite example, we find the following approximation to the largest zero:

$$a \sim \sqrt{2N} - c_1 N^{-1/6} \qquad c_1 \approx \left(3\pi/2^{7/4}\right)^{2/3}. \tag{1.10}$$

Observe that the first term agrees exactly with the result in Szegő [29], while the numerical value of the constant c_1, 1.98752…, comes close to that obtained using the Sturm comparison theorem based on the differential equation satisfied by the Hermite polynomials, 1.63329… . From this example it is apparent that the Coulomb fluid method does not give the correct constant in the second term in the asymptotic expansion of the extreme zeros of orthogonal polynomials. The correct value for c_1 is $6^{-1/3} i_1/2^{1/6}$, where i_1 is the positive smallest zero of the Airy function.

The 'edge' or uniform asymptotics of the Hermite polynomials is obtained in the limits, $N \to \infty$, $x \to \sqrt{2N}$ and such that $N^{1/6}(\sqrt{2N} - x)$ remains bounded. Specifically we let

$$x = \sqrt{2N} - c\,t\,N^{-1/6}.$$

From differential equation techniques [29, equation (8.22.13)] it is known that

$$\mathrm{e}^{-x^2/2}\, p_N(x) = 3^{1/3}\, \pi^{-3/4} 2^{(2N+1)/4} \sqrt{N!} N^{-1/2}\, [\mathrm{Ai}(t) + \mathrm{O}(N^{-2/3})]$$

holds with $c = 2^{-1/2} 3^{-1/3}$. Here the limit denotes the double limits indicated above. Indeed we recognize that when the density behaves as in (1.8) as $x \to b^-$, the appropriate parameter is

$$t := c[G(b)]^{3/2}(b - x). \tag{1.11}$$

Based on the above observation, we conjecture that the following universal formula holds for the edge asymptotics for a class of orthogonal polynomials.

Conjecture 1. If, as $x \to b^-$ the density σ *satisfies*

$$\sigma(x) \approx G(b)\sqrt{b-x} \tag{1.12}$$

then

$$\sqrt{w(x)}\, p_N(x) \sim d_N \mathrm{Ai}(t) \tag{1.13}$$

where t is defined by (1.11) and d_N is a constant depending only on N and $Ai(t)$ is the Airy function.

The above class of orthogonal polynomials, namely the class for which (1.12) holds, contains at least the Freud weights

$$w(x) := \exp(-|x|^\alpha) \qquad \alpha > 0 \tag{1.14}$$

and orthogonal polynomials whose weight function is of the form $\mathrm{e}^{-Q(x)}$, where $\log Q(x)/\log|x|$ has limits as $x \to \pm\infty$.

There is also a class of orthogonal polynomials which arises from the double scaling limit in certain problems in the theory of two-dimensional quantum gravity [3], whose density function, $\sigma(x)$, has the following edge behaviour:

$$\sigma(x) \sim A_k\, b^{1/2-k}\, (b-x)^{k+1/2} \qquad x \approx b \quad k = 0, 1, 2, \ldots.$$

Here A_k is a positive constant depending on k and $b = B_k\sqrt{N}$, is positive. Using the procedure described above (see also [5, 4]) we find that the largest zero in this case is

$$a \sim b - C_k\, b^{(2k-1)/(2k+3)}$$

where C_k has the approximate value

$$C_k \approx \left(\frac{2k+3}{2A_k}\right)^{2/(2k+3)}.$$

By introducing a scaling variable which generalizes the previous t, we expect the asymptotic relation

$$\sqrt{w(x)}\, p_N(x) \sim D_N\, \Psi(s) \tag{1.15}$$

with

$$s := A_k^{2/(2k+3)}\, b^{(1-2k)/(2k+3)}(b-x)$$

to hold, where $\Psi(s)$ is a solution of the following differential equation:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2} + q(s)\right) y(s) = 0$$

and the potential $q(s)$ having the large-s behaviour

$$q(s) \sim s^{2k+1} + \mathrm{O}(s^{2k}) \qquad \text{as } s \to \infty. \tag{1.16}$$

In this paper we employ the Coulomb fluid approximation to determine the large-N behaviour of the largest zeros of the Wilson polynomials [31], q-Laguerre polynomials [25, 13] and q^{-1}-Hermite polynomials [1, 16]. The Wilson polynomials are treated in section 2 and the limiting behaviour of their zeros resembles that of the Laguerre polynomials (Szegő [29]). The zeros of the q-Laguerre polynomials are well separated, i.e. the ratio of two consecutive zeros is at least q [25], and this is reflected in the asymptotic formula of their largest zeros which is derived in section 3. Our asymptotic result even exhibits the correct limiting behaviour as $q \to 1^-$. In section 4 the large-degree behaviour of the largest zeros of the q^{-1}-Hermite polynomials is analysed and again has the correct behaviour as

$q \to 1^-$. In section 5 we use chain sequences [10, 15] to give sharp upper bounds for the largest zeros of q-Laguerre and q^{-1}-Hermite polynomials.

Section 6 contains large-degree asymptotics of the largest zeros of the polynomials orthogonal with respect the weight functions

$$w_m(x) := \exp\left[\frac{-(\ln x)^m}{m(-\ln q)^{m-1}}\right] \tag{1.17}$$

where m is an even positive integer, $q \in (0, 1)$ and $x \in (0, \infty)$. The asymptotics are carried out using the Coulomb fluid method. Lubinsky and Sharif [21] have already determined the first term in this asymptotic expansion. Our determination of the second term gave the surprising result that the case $m = 2$ is very different from the rest of the cases $m > 0$, but $m \neq 2$.

We shall call these weights 'the weak exponential weights', for lack of a better name. Such a class of weight function for $m = 2$ arises from the study of electronic transport in disordered solids [8].

The Coulomb fluid method is not mathematically rigorous but seems to be very powerful and accurate. It may be appropriate here to quote Dyson's description of the Coulomb fluid method [11, page 158]:

> These assumptions . . . can be summarized in the single statement that for large N the Coulomb gas obeys the laws of classical thermodynamics. The assumption . . . means that the free energy density at any point being a function of the local density and temperature alone. To a physicist these assumptions are so hallowed by custom that they hardly require justification . . .

A birth and death process gives rise to a sequence of orthogonal polynomials $\{F_n(x)\}$ generated by

$$\mu_{n+1} F_{n+1}(x) = [\lambda_n + \mu_n - x]\ F_n(x) - \lambda_{n-1} F_{n-1}(x) \tag{1.18}$$

with

$$F_0(x) := 1,\ F_1(x) = [\lambda_0 + \mu_0 - x]/\lambda_0. \tag{1.19}$$

Orthogonal polynomials that arise from birth and death processes have their zeros in $(0, \infty)$. In [7] we reformulated a powerful theorem of Maté *et al* [22] and established the following theorem.

Theorem 1.1. Let $\{F_n\}$ be a family of birth and death process polynomials satisfying (1.18) and (1.19) and assume

$$\lambda_n = a^2\, n^{2\delta}[1 + \mathrm{o}(n^{-2/3})] \tag{1.20}$$

and

$$\mu_n = a^2\, n^{2\delta}[1 + \mathrm{o}(n^{-2/3})] \tag{1.21}$$

as $n \to \infty$ and assume $\delta > 0$. Let the zeros of F_n be arranged as

$$X_{n,1} > X_{n,2} > \cdots > X_{n,n} > 0 \tag{1.22}$$

and assume $\mu_0 = 0$. Furthermore, assume that $\{i_n\}_1^\infty$ are the positive zeros of the Airy function arranged in increasing order. Then we have

$$\sqrt{X_{n,k}} = 2\, a\, n^\delta\ \left[1 - \tfrac{1}{2}\delta^{2/3}\, 3^{-1/3} i_k n^{-2/3} + \mathrm{o}(n^{-2/3})\right] \tag{1.23}$$

or equivalently

$$\sqrt{X_{n,k}} = \sqrt{2\,(\lambda_n + \mu_n)}\, \left[1 - \frac{1}{2}\delta^{2/3}\, 3^{-1/3} i_k \left(\frac{\lambda_n + \mu_n}{2a^2}\right)^{-1/(3\delta)} + \mathrm{o}(n^{-2/3})\right]. \tag{1.24}$$

This theorem is relevant to our treatment of the zeros of the Wilson polynomials in section 2.

2. Wilson polynomials

The Wilson polynomials are orthogonal with respect to the weight function

$$w(x, a_1, a_2, a_3, a_4) = \frac{\prod_{j=1}^4 \Gamma(a_j + \mathrm{i}\sqrt{x})\Gamma(a_j - \mathrm{i}\sqrt{x})}{\Gamma(2\mathrm{i}\sqrt{x})\Gamma(-2\mathrm{i}\sqrt{x})} \qquad x \in [0, \infty) \quad a_j \in \mathbb{R}. \tag{2.1}$$

For the Wilson weight function we find

$$u'(x) = -\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\mathrm{i}}{\sqrt{x}}[\psi(2\mathrm{i}\sqrt{x}) - \psi(-2\mathrm{i}\sqrt{x})] - \frac{\mathrm{i}}{\sqrt{x}}\sum_{j=1}^4[\psi(a_j + \mathrm{i}\sqrt{x}) - \psi(a_j - \mathrm{i}\sqrt{x})]$$

$$= \sum_{j=1}^4\sum_{n=0}^\infty \frac{1}{x + (a_j + n)^2} - \sum_{n=0}^\infty \frac{1}{x + n^2/4} \tag{2.2}$$

where $\psi(\cdot)$ is the di-Gamma function which has the representation

$$\psi(x) - \psi(y) = \sum_{n=0}^\infty\left[\frac{1}{n+y} - \frac{1}{n+x}\right].$$

The solution to the integral equation (1.4) is

$$\sigma(x) = \frac{1}{2\pi^2} P\int_0^b \frac{1}{y-x}\sqrt{\frac{y}{b-y}}\sum_{n=0}^\infty\left(\sum_{j=1}^4 \frac{1}{y + (a_j+n)^2} - \frac{1}{y + n^2/4}\right). \tag{2.3}$$

An integration using

$$P\int_0^b \frac{\mathrm{d}y}{y-x}\sqrt{\frac{y}{b-y}}\,\frac{1}{y+c} = \frac{\pi c}{b+c}\,\frac{1}{x+c} \qquad \text{for } c > 0$$

gives the following representation for σ:

$$\sigma(x) = \frac{1}{2\pi}\sqrt{\frac{b-x}{x}}\sum_{n=0}^\infty\left[\sum_{j=1}^4 \frac{1}{x + (a_j+n)^2}\,\frac{a_j + n}{\sqrt{b + (a_j+n)^2}} - \frac{n/2}{\sqrt{b + n^2/4}}\,\frac{1}{x + n^2/4}\right] \tag{2.4}$$

for $x \in (0, b)$. Approximating the sum in (2.4) by an integral, we find, using Mathematica, that $\sigma(x)$ is approximately

$$\sigma(x) \sim \frac{1}{2\pi\sqrt{x}}\left(\frac{1}{2}\sum_{j=1}^4 \ln\left[\frac{2(b-x) + 2\sqrt{b + a_j^2} + x + a_j}{x + a_j^2}\right] - \ln\left[\frac{2b + 2\sqrt{b(b-x)}}{x}\right]\right). \tag{2.5}$$

The normalization condition in (1.1) becomes

$$N = \int_0^b \sigma(x)\,\mathrm{d}x = \frac{1}{2}\left[\sum_{j=1}^4\sum_{n=0}^\infty\left(1 - \frac{a_j + n}{\sqrt{b + (a_j+n)^2}}\right) - \sum_{n=0}^\infty\left(1 - \frac{n/2}{\sqrt{b + n^2/4}}\right)\right]$$

$$\approx \frac{1}{2}\left[\sum_{j=1}^4\int_{a_j}^\infty\left(1 - \frac{t}{\sqrt{b^2 + t^2}}\right)\mathrm{d}t - 2\int_0^\infty\left(1 - \frac{t}{\sqrt{b + t^2/4}}\right)\mathrm{d}t\right]$$

$$= \frac{1}{2}\left[\sum_{j=1}^4\left(\sqrt{b + a_j^2} - a_j\right) - 2\sqrt{b}\right]. \tag{2.6}$$

In other words N and b are related through the algebraic equation

$$2\left(N + \frac{1}{2}\sum_{j=1}^{4} a_j\right) = \sum_{j=1}^{4}\sqrt{b + a_j^2} - 2\sqrt{b}.$$

This gives the following asymptotic form of the dependence of b on N, and for sufficiently large b and N we find

$$b = \left(N + \frac{1}{2}\sum_{j=1}^{4} a_j\right)^2 + \mathrm{O}(1/N). \tag{2.7}$$

We obtain $G(b)$ for the largest zero, by expanding the density near b

$$G(b) = \frac{1}{2\pi\sqrt{b}}\left(\sum_{j=1}^{4}\frac{1}{\sqrt{b + a_j^2}} - \frac{2}{\sqrt{b}}\right) \sim \frac{1}{\pi b}. \tag{2.8}$$

Thus the largest zero of the Wilson polynomials obeys the asymptotic approximation

$$a \sim \left(N + \sum_{j=1}^{4} a_j\right)^2 - c_2\left(N + \frac{1}{2}\sum_{j=1}^{4} a_j\right)^{4/3} \tag{2.9}$$

and

$$c_2 \approx (3\pi/2)^{2/3}. \tag{2.10}$$

As we saw in (1.11), the Coulomb fluid approximation for the Hermite polynomials gave

$$c_1 = (3\pi/2^{7/4})^{2/3}$$

while the correct value is

$$c_1 = i_1\, 6^{-1/3}\, 2^{-1/6} \tag{2.11}$$

and i_1 is the positive smallest zero of the Airy function. This suggests that the Coulomb fluid approximation reads $6^{-1/3} i_1$ as $(3\pi)^{2/3}/2$. With this analogy it is plausible that

$$c_2 = 3^{-1/3} i_1. \tag{2.12}$$

This would lead to

$$a = \left(N + \sum_{j=1}^{4} a_j\right)^2 - 3^{-1/3} i_1\left(N + \frac{1}{2}\sum_{j=1}^{4} a_j\right)^{4/3} + \mathrm{o}(N^{4/3}). \tag{2.13}$$

The Wilson polynomials [31] come from a birth and death process with

$$\lambda_n = \frac{(a_1 + a_2 + n)(a_1 + a_3 + n)(a_1 + a_4 + n)(s + n - 1)}{(s + 2n - 1)(s + 2n)} \tag{2.14}$$

$$\mu_n = \frac{n(a_2 + a_3 + n)(a_2 + a_4 + n)(a_3 + a_4 + n)}{(s + 2n - 1)(s + 2n - 2)} \tag{2.15}$$

for $n \geqslant 0$, where a_1, a_2, a_3, a_4 are positive parameters and

$$s := \sum_{j=1}^{4} a_j. \tag{2.16}$$

Thus $a = \frac{1}{2}$, $\delta = 1$ in (1.20) and (1.21), and for $X_{n,k}(W)$, the zeros of the Wilson polynomials, we obtain

$$X_{n,k}(W) = (n + s)^2 - 3^{-1/3}(n + s)^{4/3} i_k + \mathrm{o}(n^{4/3}). \tag{2.17}$$

In [7] we conjectured that

$$\sqrt{X_{n,k}(W)} = (n+s) - \tfrac{1}{2}\, 3^{-1/3}(n+s)^{1/3}\,\{i_k + \epsilon_n\} \tag{2.18}$$

where ϵ_n is positive for all $n,\ n > 0$ and $\epsilon_n \to 0$ as $n \to \infty$. Of course, our asymptotic result (2.16) shows that $\epsilon_n \to 0$ as $n \to \infty$.

3. q-Laguerre polynomials

The q-Laguerre polynomials were introduced by Hahn and their Hamburger moment problem was investigated by Moak [25] and completed by Ismail and Rahman in [17]. They are orthogonal with respect to the weight function

$$w(x) := \frac{1}{(-(1-q)x;q)_\infty} \qquad x \in (0,\infty) \tag{3.1}$$

where $0 < q < 1$ and the q-shifted factorials are

$$(\alpha;q)_0 := 1 \qquad (\alpha;q)_n := \prod_{k=1}^{n}(1-\alpha q^{k-1}) \quad n = 1, 2, \ldots \text{ or } \infty \tag{3.2}$$

(see [13]). Note that this is related to an indeterminate moment problem for which the four entire functions that gives the Nevanlinna parametrization of the measures of orthogonality have been found recently, [25, 17]. We shall determine the largest zero of the q-Laguerre polynomials using the technique mentioned in section 1. The q-Laguerre polynomials are also of interest in certain physical applications [9].

First we find

$$u'(x) := -\frac{w'(x)}{w(x)} = \sum_{n=0}^{\infty}\frac{1}{x+a_n} \tag{3.3}$$

where

$$a_n := \frac{1}{(1-q)q^n}. \tag{3.4}$$

We also use the parametrization

$$q = \mathrm{e}^{-\beta} \qquad \text{with } 0 < \beta < \infty. \tag{3.5}$$

The function $\sigma(x)$ now reads

$$\sigma(x) = \frac{1}{2\pi}\sqrt{\frac{b-x}{x}}\sum_{n=0}^{\infty}\sqrt{\frac{a_n}{a_n+b}}\,\frac{1}{a_n+x} \qquad 0 < x < b. \tag{3.6}$$

The function $G(b)$ is

$$\begin{aligned} G(b) &= \frac{1}{2\pi\sqrt{b}}\sum_{n=0}^{\infty}\frac{\sqrt{a_n}}{(a_n+b)^{3/2}} \\ &\approx \frac{1}{2\pi\beta\sqrt{b}}\int_{1/(1-q)}^{\infty}\frac{\mathrm{d}u}{\sqrt{u}(u+b)^{3/2}} \\ &= \frac{1}{\pi\beta b^{3/2}}\lim_{R\to\infty}\left[\sqrt{\frac{R}{b+R}} - \sqrt{\frac{1}{1+(1-q)b}}\right]. \end{aligned}$$

In the above computation we have replaced the sum over n by an integral. Thus

$$G(b) \approx \frac{1}{\pi\beta b^{3/2}}\left[1 - \sqrt{\frac{1}{1+(1-q)b}}\right]. \tag{3.7}$$

We now determine b as a function of N. From the normalization condition in (1.1) we obtain the following from (3.6):

$$\begin{aligned}
N &= \int_0^b \sigma(x)\,\mathrm{d}x = \frac{1}{2}\sum_{n=0}^{\infty}\left(1 - \sqrt{\frac{a_n}{a_n+b}}\right) \\
&\approx \frac{1}{2\beta}\int_1^{\infty}\frac{\mathrm{d}u}{u}\left(1 - \sqrt{\frac{u}{u+(1-q)b}}\right) \\
&= \lim_{R\to\infty}\left[\ln\frac{R}{(\sqrt{R}+\sqrt{R+(1-q)b})^2} + 2\ln(1+\sqrt{1+(1-q)b})\right] \\
&= \frac{1}{\beta}\ln\left(\frac{1+\sqrt{1+(1-q)b}}{2}\right).
\end{aligned}$$

Thus

$$b = \frac{4(q^{-2N}-q^{-N})}{1-q}. \tag{3.8}$$

Note that we recover the edge parameter for the Laguerre polynomials in the limit $q \to 1^-$. Indeed we have

$$\lim_{q\to 1^-} b = 4N. \tag{3.9}$$

Equations (1.9) and (3.7) give

$$a \approx b - (3\pi\beta/2)^{2/3}\left(1 - \frac{1}{\sqrt{1+b(1-q)}}\right)^{-2/3} b. \tag{3.10}$$

Observe that in this case the correction term is of the same order as the main term. This seems to be the typical behaviour when the zeros have exponential growth or when the weight function behaves like $\exp(-c|\log x|^{\alpha})$, for $c > 0$ and $\alpha > 0$. One possible explanation is that the Coulomb fluid method gives b as function of N, say $b \approx C\exp(f(N))$. The next approximation (1.7) basically changes N to $N + h(N)$, say where $h(N) = \mathrm{o}(N)$. This may have the effect of only changing the multiplicative constant C. For example with $f(N) = N$, $g(N) = c$, a constant, the effect of the second approximation will be to replace C by $C\mathrm{e}^c$.

We now use a different approximation of the sum in the equation following (3.7). We utilize the q-integral [13]

$$\int_0^a f(x)\,\mathrm{d}_q x := \sum_{n=0}^{\infty} f(aq^n)\,(aq^n - aq^{n+1}). \tag{3.11}$$

The q-integral is just an infinite Riemann sum using the evaluation points $\{aq^n : 0 \leqslant n < \infty\}$. We go back to the equation defining N and proceed as follows:

$$N = \int_0^b \sigma(x)\,\mathrm{d}x = \frac{1}{2}\sum_{n=0}^{\infty}\left[1 - \frac{1}{\sqrt{1+b(1-q)q^n}}\right]$$

$$= \frac{1}{2(1-q)} \sum_{n=0}^{\infty} \left[1 - (1 + b(1-q)q^n)^{-1/2}\right] \frac{q^n(1-q)}{q^n}$$

$$=: \frac{1}{2(1-q)} \int_0^1 \left[1 - (1 + b(1-q)u)^{-1/2}\right] \frac{\mathrm{d}_q u}{u}. \tag{3.12}$$

Now taking q to be near 1, we approximate the q-integral by a Lebesgue integral and obtain

$$N \approx \int_0^1 \left[1 - (1 + b(1-q)u)^{-1/2}\right] \frac{\mathrm{d}u}{u}$$

$$= \frac{1}{1-q} \int_1^{\sqrt{1+b(1-q)}} \frac{\mathrm{d}v}{1+v}.$$

Therefore

$$N \approx \ln\left[\frac{1 + (1 + b(1-q))^{1/2}}{2}\right]. \tag{3.13}$$

This results in the following approximation to b:

$$b \approx 4\mathrm{e}^{N(1-q)} \, \frac{\mathrm{e}^{N(1-q)} - 1}{1-q}. \tag{3.14}$$

The quantity $G(b)$ can also be evaluated in the same approximation:

$$G(b) = \frac{1}{2\sqrt{b}} \int_0^1 \frac{\mathrm{d}_q u}{[1 + b(1-q)u]^{3/2}}$$

and we obtain

$$G(b) \approx \frac{1}{\pi b^{3/2}(1-q)} \left[1 - (1 + b(1-q))^{-1/2}\right]. \tag{3.15}$$

Note that this is same as that obtained previously by approximating the sum as an integral provided we identify β with $1-q$ for q is close to 1.

Thus for sufficiently large b and q close to 1, we have

$$a \approx b - \left[\frac{3\pi(1-q)}{2(1 - (1 + b(1-q))^{-1/2})}\right]^{2/3} b \tag{3.16}$$

with b given by (3.8) or (3.14).

We remark here that the edge parameter, b, obtained by the first method is larger than that obtained using the second method, both of which compare favourably with the chain sequence estimate to be presented in section 5. This is so since $\mathrm{e}^{N(1-q)}(\mathrm{e}^{N(1-q)} - 1) < q^{-2N} - q^{-N}$ if and only if $(\mathrm{e}^{N(1-q)} - q^{-N})(\mathrm{e}^{N(1-q)} + q^{-N} - 1) < 0$, which is equivalent to $\mathrm{e}^{N(1-q)} < q^{-N}$, namely $q\ \mathrm{e}^{1-q} < 1$. The last inequality is $1 - \mathrm{e}^{-\beta} < \beta$ and obviously holds for all $\beta > 0$.

Here again the magic constant $(3\pi)^{2/3}$ appears and the correct value may be that we replace $(3\pi)^{2/3}$ by $2(6^{-1/3})i_1$ but further numerical evidence is needed.

4. q^{-1}-Hermite polynomials

Ismail and Masson [16] proved that the q^{-1}-Hermite polynomials are orthogonal with respect to the weight function

$$w(x;\eta) := \frac{\mathrm{e}^{2\eta_1} \sin\eta_2 \cosh\eta_1 (q, -q\mathrm{e}^{2\eta_1}, -q\mathrm{e}^{-2\eta_1}; q)_\infty |(q\mathrm{e}^{2i\eta_2}; q)_\infty|^2}{\pi \ |(\mathrm{e}^{\xi+\eta}, -\mathrm{e}^{\eta-\xi}, -q\mathrm{e}^{\xi-\eta}, q\mathrm{e}^{-\xi-\eta}; q)_\infty|^2} \tag{4.1}$$

where

$$x = \sinh\xi \qquad \eta = \eta_1 + \mathrm{i}\eta_2 \quad \eta_1 \in (-\infty, \infty) \quad 0 < \eta_2 < \pi/2. \tag{4.2}$$

This is an interesting example because the weight function depends on the two parameters η_1 and η_2, while the orthogonal polynomials, and hence their zeros, are independent of these parameters.

In this case

$$\sigma(x) = \frac{1}{2\pi^2}\sqrt{\frac{b-x}{x+b}}P\int_{-b}^{b}\frac{-w'(y;\eta)}{w(y;\eta)(y-x)}\sqrt{\frac{y+b}{b-y}}\,\mathrm{d}y. \tag{4.3}$$

As in the case of q-Laguerre polynomials we set

$$q = \mathrm{e}^{-\beta} \tag{4.4}$$

and apply

$$(\mathrm{e}^{\xi+\eta}, -\mathrm{e}^{\eta-\xi}; q)_\infty = \prod_{n=0}^{\infty}\left[1 - 2q^n\mathrm{e}^{\eta}x - q^{2n}\mathrm{e}^{2\eta}\right]$$

$$(-q\mathrm{e}^{\xi-\eta}, q\mathrm{e}^{-\xi-\eta}; q)_\infty = \prod_{n=0}^{\infty}\left[1 + 2q^{n+1}\mathrm{e}^{-\eta}x - q^{2n+2}\mathrm{e}^{-2\eta}\right]$$

to find from (4.1) that, for real x, we have

$$\begin{aligned}-\frac{w'(x;\eta)}{w(x;\eta)} &= \sum_{n=0}^{\infty}\left[\frac{1}{x-\sinh(n\beta-\eta)} + \frac{1}{x+\sinh((n+1)\beta+\eta)}\right] + (\eta\to\overline{\eta})\\ &= \sum_{-\infty}^{\infty}\left[\frac{1}{x-\sinh(n\beta-\eta)} + \frac{1}{x-\sinh(n\beta-\overline{\eta})}\right].\end{aligned} \tag{4.5}$$

From (4.3) and (4.5) it is easy to obtain

$$\sigma(x) = \frac{1}{2\pi^2}\sqrt{\frac{b-x}{x+b}}\sum_{-\infty}^{\infty}P\int_{-b}^{b}\frac{y+b}{\sqrt{b^2-y^2}}\left[\frac{1}{y-\sinh(n\beta-\eta)} + \eta\to\overline{\eta}\right]\frac{\mathrm{d}y}{y-x}. \tag{4.6}$$

If $\mathrm{Im}\,a \neq 0$ then

$$\begin{aligned}P\int_{-b}^{b}\frac{1}{y-a}\frac{y+b}{\sqrt{b^2-y^2}}\frac{\mathrm{d}y}{y-x} &= P\int_{-b}^{b}\frac{1}{\sqrt{b^2-y^2}}\left[1+\frac{a+b}{y-a}\right]\frac{\mathrm{d}y}{y-x}\\ &= (a+b)P\int_{-b}^{b}\frac{1}{(y-a)\sqrt{b^2-y^2}}\frac{\mathrm{d}y}{y-x}\\ &= \frac{a+b}{x-a}P\int_{-b}^{b}\left[\frac{1}{y-x} - \frac{1}{(y-a)}\right]\frac{\mathrm{d}y}{\sqrt{b^2-y^2}}\\ &= \frac{a+b}{x-a}\int_{-b}^{b}\frac{\mathrm{d}y}{(a-y)\sqrt{b^2-y^2}}\\ &= \frac{\pi}{x-a}\sqrt{\frac{a+b}{a-b}}.\end{aligned}$$

Thus we have proved

$$P\int_{-b}^{b}\frac{1}{y-a}\frac{y+b}{\sqrt{b^2-y^2}}\frac{\mathrm{d}y}{y-x}=\frac{\pi}{x-a}\sqrt{\frac{a+b}{a-b}}. \tag{4.7}$$

Therefore

$$\sigma(x)=\frac{1}{2\pi}\sqrt{\frac{b-x}{x+b}}\;\sum_{-\infty}^{\infty}\left[\sqrt{\frac{b+\sinh(n\beta-\eta)}{\sinh(n\beta-\eta)-b}}\;\frac{1}{x-\sinh(n\beta-\eta)}+\eta\to\overline{\eta}\right]. \tag{4.8}$$

We similarly evaluate $\int_{-b}^{b}\sigma(x)\,\mathrm{d}x$ and the result is

$$N=\int_{-b}^{b}\sigma(x)\;\mathrm{d}x=\frac{1}{2}\sum_{-\infty}^{\infty}\left[1-\sqrt{\frac{b+\sinh(n\beta-\eta)}{\sinh(n\beta-\eta)-b}}\right]+\eta\to\overline{\eta}. \tag{4.9}$$

We now approximate the sums in (4.9) by integrals, i.e. we apply

$$\begin{aligned}\sum_{-\infty}^{\infty}\left[1-\sqrt{\frac{b+\sinh(n\beta-\eta)}{\sinh(n\beta-\eta)-b}}\right]&\approx\frac{1}{\beta}\int_{-\infty}^{\infty}\left[1-\sqrt{\frac{b+\sinh(u-\eta)}{\sinh(u-\eta)-b}}\right]\mathrm{d}u\\&=\frac{1}{\beta}\int_{-\infty-\mathrm{i}\eta_2}^{\infty-\mathrm{i}\eta_2}\left[1-\sqrt{\frac{b+\sinh u}{\sinh u-b}}\right]\mathrm{d}u\end{aligned} \tag{4.10}$$

where η_2 is as in (4.2). We also have another sum as in (4.10) but η is replaced by $\overline{\eta}$, hence the resulting integral has η_2 replaced by $-\eta_2$. After a change of variable $u\to -u$ we can combine the two approximating integrals and establish

$$\begin{aligned}\beta N&\approx\int_{-\infty-\mathrm{i}\eta_2}^{\infty-\mathrm{i}\eta_2}\left[1-\frac{\sinh u}{\sqrt{\sinh^2 u-b^2}}\right]\mathrm{d}u\\&=\lim_{R\to\infty}\left[u-\cosh^{-1}\left(\frac{\cosh u}{\sqrt{1+b^2}}\right)\right]_{-R-\mathrm{i}\eta_2}^{R-i\eta_2}\\&=\ln(1+b^2).\end{aligned} \tag{4.11}$$

This shows that

$$b\approx\sqrt{q^{-N}-1}. \tag{4.12}$$

Observe that the estimate (4.12) is independent of η_1 and η_2, as it should be. From the three-term recurrence relations of the q^{-1}-Hermite polynomials $\{h_n(x|q)\}$ in [1] or [16]

$$2xh_n(x|q)=h_{n+1}(x|q)+q^{-n}(1-q^n)h_{n-1}(x|q) \tag{4.13}$$

the recurrence relation of the Hermite polynomials $\{H_n(x)\}$ in [29], for example

$$2xH_n(x)=H_{n+1}(x)+2n\,H_{n-1}(x) \tag{4.14}$$

and the initial conditions $H_0(x)=h_0(x)=1$, $H_1(x)=h_1(x|q)=2x$, it follows that

$$\lim_{q\to1^-}\frac{2^{n/2}}{(1-q)^{n/2}}h_N(x\sqrt{(1-q)/2}\;q)=H_N(x). \tag{4.15}$$

It is interesting to note that (4.12) exhibits the correct limiting behaviour as $q\to1^-$, since b in (4.12) has to be replaced by $b\sqrt{(1-q)/2}$. The result agrees with the first term in (1.11).

We will determine the behaviour near the edge after we have derived the estimate (4.12) using a different weight function. Askey [1] proved that the q^{-1}-Hermite polynomials are orthogonal with respect to the weight function

$$w(x) := \frac{(1+x^2)^{-1/2}}{\prod_{n=0}^{\infty}[1+2(2x^2+1)q^n+q^{2n}]}. \tag{4.16}$$

To apply the Coulomb fluid method to the weight function (4.16) first we note that it is easy to obtain

$$\begin{aligned} -\frac{w'(x)}{w(x)} &= \frac{x}{1+x^2} + \sum_{n=1}^{\infty} \frac{8x}{q^{-n}+2(2x^2+1)+q^n} \\ &= \frac{x}{1+x^2} + \left[\sum_{n=1}^{\infty} + \sum_{n=-1}^{-\infty}\right] \frac{4x}{q^{-n}+2(2x^2+1)+q^n} \\ &= \sum_{-\infty}^{\infty} \frac{x}{x^2+c_n^2} \end{aligned} \tag{4.17}$$

where

$$c_n = [q^{n/2} + q^{-n/2}]/2. \tag{4.18}$$

Using equation (4.7) and proceeding as in the case of the weight function (4.1), we see that

$$\sigma(x) = \frac{1}{4\pi}\sqrt{\frac{b-x}{b+x}} \sum_{-\infty}^{\infty} \left[\frac{\mathrm{i}(b-\mathrm{i}c_n)}{(x+\mathrm{i}c_n)\sqrt{b^2+c_n^2}} + \text{complex conjugate}\right] \tag{4.19}$$

and the side condition $N = \int_{-b}^{b} \sigma(x)\,\mathrm{d}x$ becomes

$$\begin{aligned} 4N &= \sum_{-\infty}^{\infty} \left[1 - \sqrt{\frac{-b+\mathrm{i}c_n}{b+\mathrm{i}c_n}} + 1 - \sqrt{\frac{b+\mathrm{i}c_n}{-b+\mathrm{i}c_n}}\right] \\ &= 2\sum_{-\infty}^{\infty} \left[1 - \frac{c_n}{\sqrt{b^2+c_n^2}}\right]. \end{aligned} \tag{4.20}$$

We again approximate the sums in (4.20) by integrals, and after calculations similar to those used in deriving (4.11) we obtain

$$\beta N \approx \int_{-\infty}^{\infty} \left[1 - \frac{\cosh u}{\sqrt{b^2+\cosh^2 u}}\right] \mathrm{d}u.$$

Therefore

$$\begin{aligned} \beta N &\approx \lim_{R\to\infty} \left[u - \sinh^{-1}\left(\frac{\sinh u}{\sqrt{1+b^2}}\right)\right]_{-R}^{R} \\ &= 2\lim_{R\to\infty} \left[R - \ln(2\sinh R) + \tfrac{1}{2}\ln(1+b^2)\right] = \ln(1+b^2). \end{aligned}$$

This is the last equality in (4.11), and hence (4.12) follows.

We can use the q-integral to approximate the sum in (4.20). Recall [13] that

$$\int_0^{\infty} f(x)\,\mathrm{d}_q x = \sum_{-\infty}^{\infty} f(q^n)\,(q^n - q^{n+1}). \tag{4.21}$$

Now equation (4.20) is

$$2(1-\sqrt{q})N = \sum_{-\infty}^{\infty} \frac{q^{n/2}-q^{(n+1)/2}}{q^{n/2}}\left[1-\frac{(q^{n/2}+q^{-n/2})/2}{\sqrt{b^2+(q^{n/2}+q^{-n/2})^2/4}}\right]$$

$$= \int_0^\infty \left[1-\frac{(u+u^{-1})/2}{\sqrt{b^2+(u+u^{-1})^2/4}}\right]\frac{\mathrm{d}_q u}{u}$$

$$\approx \int_{-\infty}^{\infty}\left[1-\frac{\cosh v}{\sqrt{b^2+\cosh^2 v}}\right]\mathrm{d}v.$$

This gives (4.11) with β replaced by $2(1-\sqrt{q})$. Therefore another estimate for b is

$$b \approx \left(\mathrm{e}^{2N(1-\sqrt{q})}-1\right)^{1/2}. \tag{4.22}$$

Thus we succeeded in replacing (4.12) by (4.22), which is a smaller estimate (see the discussion following (3.16)).

To determine the edge behaviour we go to (4.19) and find

$$G(b) = \frac{\sqrt{b/2}}{\pi}\sum_{-\infty}^{\infty}\frac{c_n}{(b^2+c_n^2)^{3/2}}$$

$$\approx \frac{\sqrt{b/2}}{\pi(1-\sqrt{q})}\int_{-\infty}^{\infty}\frac{\cosh v\,\mathrm{d}v}{[b^2+\cosh^2 v]^{3/2}}$$

$$= \frac{\sqrt{2b}}{\pi(1-\sqrt{q})}\int_0^\infty \frac{\mathrm{d}w}{[b^2+1+w^2]^{3/2}}.$$

Thus

$$G(b) \approx \frac{\sqrt{2b}}{\pi(1-\sqrt{q})}\,\frac{1}{1+b^2}. \tag{4.23}$$

Substituting for $G(b)$ from (4.23) in (1.9) we obtain

$$a \approx b - \left(\frac{3\pi(1-\sqrt{q})\,(1+b^2)}{2\sqrt{2b}}\right)^{2/3}. \tag{4.24}$$

Therefore

$$a \approx \left[1-\tfrac{1}{2}\left(3\pi(1-\sqrt{q})\right)^{2/3}\right]\left[\mathrm{e}^{2N(1-\sqrt{q})}-1\right]^{1/2}. \tag{4.25}$$

The limit $q \to 1^-$ of (4.24) is interesting. As was pointed out earlier, we need to change variables as in (4.13), so a needs to be replaced by $a\sqrt{(1-q)/2}$. Thus equation (4.24) becomes

$$a \approx \left[\frac{2(\mathrm{e}^{2N(1-\sqrt{q})}-1)}{1-q}\right]^{1/2} - c_4\frac{\mathrm{e}^{4N(1-\sqrt{q})/3}\,(1-\sqrt{q})^{2/3}}{\left(\mathrm{e}^{2N(1-\sqrt{q})}-1\right)^{1/6}\,\sqrt{(1-q)/2}} \tag{4.26}$$

$$c_4 \approx \left(\frac{3\pi}{2\sqrt{2}}\right)^{2/3}. \tag{4.27}$$

As $q \to 1^-$ the right-hand side of (4.25) reduces to the correct limit as given by the right-hand side of (1.11).

Here again c_4 contains our old friend $(3\pi)^{2/3}$ which indicates that it may be replaced by an algebraic multiple of i_1. We think it may be that

$$c_4 = 6^{-1/3} i_1 \tag{4.28}$$

although this will be very surprising, if it holds true. In the area of random matrix models it is believed that there is a universal law governing the behaviour of the density at the tail of the eigenvalue spectrum (point spectrum). Our work gives an interpretation of the tail behaviour in terms of the largest zeros of orthogonal polynomials. We propose that this universality principle broadly states that the constant in the second term of the asymptotic development of the orthogonal polynomials is the first positive zero of the Airy function multiplied by an algebraic number. Some assumptions on the recursion coefficients or the analytic properties of the the weight functions are required. This universality principle is probably true for orthogonal polynomials whose recurrence coefficients are monotone and have algebraic growth. More precisely we believe the following to hold true.

Conjecture 2. Let $p_n(x)$ be symmetric orthogonal polynomials generated via

$$p_0(x) := 1 \qquad p_i(x) = x/a_0 \qquad x p_n(x) = a_n p_{n+1}(x) + a_{n-1} p_{n-1}(x) \tag{4.29}$$

where $\{a_n\}$ is a monotonically increasing sequence and

$$a_n = c\, n^{\gamma}(1 + \mathrm{o}(1)) \qquad \gamma > 0. \tag{4.30}$$

Then the largest zero of $p_n(x)$ satisfies

$$x_{N,1} = 2c\, n^{\gamma}\left[1 + \xi i_1 n^{\delta} + \mathrm{o}(n^{\delta})\right] \tag{4.31}$$

where $0 > \delta$ and ξ is an algebraic number and i_1 is the smallest positive zero of the Airy function.

The number δ in conjecture 2 depends on δ. The recurrence coefficients of the q^{-1}-Hermite polynomials are of exponential growth. If equation (4.28) is true, then the q^{-1}-Hermite polynomials will also obey this universal law, a very surprising fact indeed.

5. Bounds using chain sequences

In this section we use chain sequences to establish sharp upper bounds for the largest zeros of the Wilson, q^{-1}-Hermite and q-Laguerre polynomials. The method is based on explicitly knowing the coefficients in the three-term recurrence relation satisfied by the polynomials.

Recall that a sequence $\{a_n : 0 < n < N\}$ is a *chain sequence* if there is a parameter sequence g_n, such that

$$a_n = g_n(1 - g_{n-1}) \qquad 0 \leqslant g_0 < 1 \quad 0 < g_n < 1 \quad 0 < n < N. \tag{5.1}$$

For detailed information, see [10] and [15]. Here N may be finite or infinite. The sequence $\{1/4\}$ is a chain sequence with $g_n = 1/2$, so is any non-negative sequence bounded above by a chain sequence.

Theorem 5.1. Assume that $Q_n(z)$ is a sequence of monic orthogonal polynomials generated by

$$Q_0(x) = 1, \quad Q_1(x) = x - \alpha_0, \quad Q_{n+1}(x) = (x - \alpha_n)Q_n(x) - \beta_n Q_{n-1}(x) \tag{5.2}$$

and let

$$B := \max\{x_n : 0 < n < N\} \qquad A := \min\{y_n : 0 < n < N\} \tag{5.3}$$

where x_n and y_n, $x_n \geqslant y_n$, are the roots of

$$(x - \alpha_n)(x - \alpha_{n-1})a_n = \beta_n \tag{5.4}$$

and a_n is any chain sequence. Then all the zeros of $Q_n(z)$ lie in (A, B).

This is [15, theorem 2] and is a restatement of the Wall–Wetzel theorem. A consequence of theorem 5.1 is the following.

Theorem 5.2. If the zeros of $Q_N(x)$ are less (greater) than A (B), then the sequence $\{\beta_n/(\alpha_n - A)(\alpha_{n-1} - A) : 0 < n < N\}$ $(\{\beta_n/(B - \alpha_n)(B - \alpha_{n-1}) : 0 < n < N\})$ is a chain sequence.

In the case of q^{-1}-Hermite polynomials $\alpha_n = 0$ and $\beta_n = q^{-n}(1 - q^n)/4$. Let $X_{N,1}(H)$ and $X_{N,1}(qH)$ be the largest zero of Hermite and q^{-1}-Hermite polynomials of degree N, respectively. With $a_n = \frac{1}{4}$ in (5.4) we obtain the following bound for $X_{N,1}(qH)$:

$$X_{N,1}(qH) < [q^{1-N}(1 - q^{N-1})]^{1/2} \tag{5.5}$$

since $q^{-n}(1 - q^n)/4$ increases with n.

The next theorem gives a more refined bound for $X_{N,1}(qH)$.

Theorem 5.3. We have

$$X_{N,1}(qH) < \left\{\sqrt{2N+1} - \frac{6^{-1/3} i_1}{(2N+1)^{1/6}}\right\} \sqrt{\frac{q^{1-N}(1 - q^{N-1})}{2(N-1)}} \tag{5.6}$$

where i_1 is the smallest positive zero of the Airy function.

Proof. For the Hermite polynomials $\alpha_n = 0$ and $\beta_n = n/2$, so by theorem 5.2 we can choose

$$a_n = \frac{n}{[2X_{N,1}(H) + \epsilon]^2} \qquad 0 < n < N \tag{5.7}$$

for any $\epsilon > 0$, which can be allowed to depend on N, in theorem 5.1. Since $q^{-n}(1 - q^n)/4$ is strictly increases with n then the result follows from theorem 5.1 and the inequality

$$X_{N,1}(H) < \sqrt{2N+1} - \frac{6^{-1/3} i_1}{(2N+1)^{1/6}} \tag{5.8}$$

of [29, theorem 6.32]. This completes the proof. □

It is easy to see that (5.6) is sharper than (5.5) for sufficiently large N. As was mentioned following (4.15), to obtain the correct limit as $q \to 1^-$ we need to replace $X_{N,1}(qH)$ by $X_{N,1}(qH)\sqrt{(1 - q)/2}$. With this renormalization the bound in (5.6) tends to the bound (5.8) of $X_{N,1}(H)$ as $q \to 1^-$.

We now treat the q-Laguerre polynomials $\{L_n^{(\alpha)}(x; q)\}$, [25]. The corresponding α_n's and β_n's are given by

$$\alpha_n = \frac{1 - q^n}{(1 - q)q^{2n+\alpha}} + \frac{1 - q^{n+\alpha+1}}{(1 - q)q^{2n+\alpha+1}} \tag{5.9}$$

$$\beta_n = \frac{(1 - q^n)(1 - q^{n+\alpha})}{(1 - q)^2 q^{4n+2\alpha-1}}. \tag{5.10}$$

It is easy to see that the roots of (5.4) with $a_n = \frac{1}{4}$ are monotone in n, thus theorem 5.1 shows that the zeros of the q-Laguerre polynomial $L_N^{(\alpha)}(z; q)$ are in (A_N, B_N) where A_N and B_N are the roots of

$$(x - \alpha_{N-1})(x - \alpha_{N-2}) = 4\beta_{N-1} \tag{5.11}$$

and the α's and β's are given by (5.9) and (5.10). The main term in A_N for sufficiently large N is

$$A_N = \frac{q^{-2N+1-\alpha}}{2(1-q)} \left[(1+q)(1+q^2) + \sqrt{16q^3 + (1+q)^2(1-q^2)^2}\right] + \mathrm{O}(q^{-N}). \tag{5.12}$$

Mathematica shows that the function

$$16x^3 + (1+x)^2(1-x^2)^2$$

is monotonic for $0 \leqslant x < 1$. Thus the quantity in square brackets in (5.12) increases with q and is bounded above by its value at $q = 1$, namely 8. Thus $B_N < 4q^{-2N+1-\alpha}/(1-q)$, an estimate, which in the special case $\alpha = 0$ is sharper than (3.8).

6. Weak exponential weights

In this section we study the largest zero for polynomials orthogonal with respect to weak exponential weights (1.17). Note that the Stieljes–Wigert polynomials [10] are special q-Laguerre polynomials and are orthogonal with respect to a weak exponential weight function with $m = 2$. Thus we expect the limiting behaviour of the largest zeros of the polynomials orthogonal with respect to $w_m(x)$ to resemble the largest zeros of the q-Laguerre polynomials, for example they should increase exponentially. It turned out that the qualitative behaviour of the correction terms depends on m and that the case $m = 2$ is an exceptional case.

As before, it is convenient to use the parametrization

$$q = \mathrm{e}^{-\beta} \qquad 0 < \beta < \infty. \tag{6.1}$$

The normalization condition for the density is

$$N = \frac{1}{2\pi}\int_0^b \sqrt{\frac{y}{b-y}}\, u'(y)\, \mathrm{d}y. \tag{6.2}$$

Using the fact that

$$u'(x) = \frac{1}{\beta^{m-1}} \frac{(\ln x)^{m-1}}{x}$$

we obtain

$$N = \frac{1}{2\pi\beta^{m-1}} \int_0^1 \frac{(\ln b + \ln t)^{m-1}}{\sqrt{t(1-t)}}\, \mathrm{d}t. \tag{6.3}$$

The integrand can be expanded by the binomial theorem and N is now expressed as a polynomial in $\ln b$ of degree $m-1$. Thus we at arrive at the asymptotic expansion

$$N = \frac{1}{2\beta^{m-1}} \left[(\ln b)^{m-1} - 2(m-1)\ln 2(\ln b)^{m-2} + \mathrm{O}((\ln b)^{m-3})\right] \tag{6.4}$$

as $b \to \infty$. Solving for b in terms of N, we find

$$b \sim \exp\left[(2N)^{1/(m-1)}\beta\right] = q^{-(2N)^{1/(m-1)}}. \tag{6.5}$$

This indicates that the zeros are well separated, as is the case with the zeros of the q-Laguerre polynomials [25].

We now proceed to determine the further refinement to the first crude estimate given by (6.5). To accomplish this the behaviour of $\sigma(x)$ for $x \to b^-$ is required. However, for $m > 2$, it appears that no explicit formula for $\sigma(x)$ is available. Therefore we shall only give an asymptotic formula for the quantity $G(b)$, introduced in the previous sections. The density is

$$\sigma_m(x) = \frac{1}{2\pi^2\beta^{m-1}}\sqrt{\frac{b-x}{x}}\, P\int_0^b \frac{(\ln y)^{m-1}}{\sqrt{y(b-y)}}\,\frac{\mathrm{d}y}{y-x}. \tag{6.6}$$

We used $\sigma_m(x)$ to exhibit the dependence on m. To compute the above principle value integral we use a generating function technique. It is clear that

$$\begin{aligned}\sum_{m=1}^{\infty}\sigma_m(x)\frac{\beta^{m-1}w^{m-1}}{(m-1)!} &= \frac{1}{2\pi^2}\sqrt{\frac{b-x}{x}}\, P\int_0^b \frac{\exp(w\ln y)}{\sqrt{y(b-y)}}\,\frac{\mathrm{d}y}{y-x}\\ &= \frac{1}{2\pi^2}\sqrt{\frac{b-x}{x}}\, P\int_0^b \frac{y^{w-1/2}}{\sqrt{(b-y)}}\,\frac{\mathrm{d}y}{y-x}. \end{aligned} \tag{6.7}$$

Therefore [14, equation (2.228.3)] yields

$$\sum_{m=1}^{\infty}\sigma_m(x)\frac{\beta^{m-1}w^{m-1}}{(m-1)!} = -\frac{b^{w-1}}{2\pi^2}\sqrt{\frac{b-x}{x}}B\left(-\tfrac{1}{2}, w+\tfrac{1}{2}\right){}_2F_1\left(1-w, 1; \tfrac{3}{2}; 1-x/b\right). \tag{6.8}$$

In the limit $x \to b^-$, we find

$$\sigma_m(x) \sim G_m(b)\sqrt{b-x} \qquad x \to b^-. \tag{6.9}$$

In this case the G's have the generating function

$$\begin{aligned}\sum_{m=0}^{\infty} G_{m+1}(b)\frac{\beta^m w^m}{m!} &= -\frac{b^{w-3/2}}{2\pi^2}\frac{\Gamma(-\frac{1}{2})\Gamma(w+\frac{1}{2})}{\Gamma(w+1)}\\ &= \frac{w}{(b\pi)^{3/2}}\exp(w(\ln b))\frac{\Gamma(w+\frac{1}{2})}{\Gamma(w+1)}. \end{aligned} \tag{6.10}$$

Since both $\Gamma(w+\frac{1}{2})$ and $\Gamma(w+1)$ are analytic functions of w in the neighbourhood of $w=0$ [12], then (6.10) shows that $G_1(b) = 0$ and $G_m(b)$ is a polynomial in $\ln b$ of degree $m-2$, for $m > 1$. Indeed we have

$$\frac{G_{m+2}}{(m+1)!}\beta^{m+1} = \frac{1}{\pi\, b^{3/2}}\,\frac{(\ln b)^m}{m!} + \text{lower-order terms}. \tag{6.11}$$

This gives the following asymptotics for the largest zeros of polynomials orthogonal with respect to weak exponential weights:

$$a \sim b - c_1\, b\left[\frac{\beta^{m-1}}{2\,(m-1)\,(\ln b)^{m-2}}\right]^{2/3} \qquad \text{as}\ \ b\to\infty \tag{6.12}$$

where

$$c_1 \approx (3\pi)^{2/3}. \tag{6.13}$$

Here again it is likely that $c_1 = 2(6^{-1/3})i_1$.

Observe that for $m = 2$, the correction term in (6.12) is of the same order as the first term, the same was noted in the case of the q-Laguerre and the q^{-1}-Hermite polynomials. This is not surprising since the largest zero as $N \to \infty$ is controlled by the large-x behaviour of the potential $u(x)$.

We indicate here that in order to obtain the lower-order terms in the expansion (6.11) it is useful to make use of the duplication formula of the Gamma function to write the right-hand side of (6.10) as

$$\frac{w}{\pi b^{3/2}}\ \exp(w\ \ln(\tfrac{1}{4}b))\ \frac{\Gamma(2w+1)}{\Gamma^2(w+1)}. \tag{6.14}$$

Equation (1.17.2) in [12] is

$$\log(\Gamma(1+z)) = -\gamma z + \sum_{n=2}^{\infty} \frac{(-1)^n\, \zeta(n)\, z^n}{n} \tag{6.15}$$

which implies

$$\log\left(\frac{\Gamma(1+2w)}{\Gamma^2(1+w)}\right) = 2\sum_{n=2}^{\infty} \frac{(-1)^n\,(2^{n-1}-1)}{n}\,\zeta(n)\,w^n. \tag{6.16}$$

Thus one can obtain a few of the lower-order terms in (6.12), if needed.

Acknowledgments

This work was done while MEHI was visiting Imperial College. He gratefully acknowledges the hospitality and support from both Imperial College and the University of South Florida.

References

[1] Askey R A 1989 Continuous q-Hermite polynomials when $q > 1$ *q-Series and Partitions (IMA volumes in Mathematics and Its Applications)* ed D Stanton (Berlin: Springer) pp 151–8

[2] Bohigas O and Giannoni M J 1984 Chaotic motion and random matrix theories *Mathematical and Computational Methods in Nuclear Physics (Proceedings, Granada, Spain, 1983) (Lecture Notes in Physics 209)* ed J S Dehesa *et al* (Berlin: Springer) pp 1–99

[3] Bowick M J and Brézin E 1991 Universal scaling of the tail of the density of eigenvalues in random matrix models *Phys. Lett.* **268B** 21–8

[4] Chen Y and Eriksen K J 1995 Gap formation of the α-ensemble *Int. J. Mod. Phys.* B **9** 1205–25

[5] Chen Y, Eriksen K J and Tracy C A 1995 Largest eigenvalue distribution in the double scaling limit of matrix models: a Coulomb fluid approach *J. Phys. A: Math. Gen.* **28** L207–11

[6] Chen Y and Ismail M E H 1998 Hermitian matrix ensembles and orthogonal polynomials *Stud. Appl. Math.* **100** 33–52

[7] Chen Y and Ismail M E H Some indeterminate moment problems and Freud-like weights *Constr. Approx.* to appear

[8] Chen Y, Ismail M E H and Muttalib K A 1992 A solvable random matrix models for disordered conductors *J. Phys.: Condens. Matter* **4** L417–23

Chen Y, Ismail M E H and Muttalib K A 1993 Metallic and insulating behavior in an exactly solvable random matrix model *J. Phys.: Condens. Matter* **5** 171–90

[9] Chen Y, Ismail M E H and Muttalib K A 1995 Asymptotics of basic Bessel functions and q-Laguerre polynomials *J. Comp. Appl. Math.* **54** 263–73

[10] Chihara T S 1978 *An Introduction to Orthogonal Polynomials* (New York: Gordon and Breach)

[11] Dyson F J 1962 Statistical theory of energy levels of complex systems, I *J. Math. Phys.* **3** 140–56

Dyson F J 1962 Statistical theory of energy levels of complex systems, II *J. Math. Phys.* **3** 157–65

Dyson F J 1962 Statistical theory of energy levels of complex systems, III *J. Math. Phys.* **3** 166–75

[12] Erdélyi A, Magnus W, Oberhettinger F and Tricomi G F 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)

[13] Gasper G and Rahman M 1990 *Basic Hypergeometric Series* (Cambridge: Cambridge University Press)

[14] Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series and Products* 4th edn (New York: Academic) p 290

[15] Ismail M E H and Li X 1992 Bounds on the extreme zeros of orthogonal polynomials *Proc. Am. Math. Soc.* **115** 131–140

[16] Ismail M E H and Masson D R 1994 q-Hermite polynomials, biorthogonal rational functions and q-beta integrals *Trans. Am. Math. Soc.* **346** 61–116

[17] Ismail M E H and Rahman M The q-Laguerre polynomials and related moment problems *J. Math. Anal. Appl.* to appear

[18] Lubinsky D S 1987 A survey of general orthogonal polynomials for weights on finite and infinite intervals *Acta Appl. Math.* **10** 237–96

[19] Lubinsky D S 1993 An update on orthogonal polynomials and weighted approximation on the real line *Acta Appl. Math.* **33** 121–64

[20] Lubinsky D S, Mhaskar H N and Saff E B 1988 A proof of Freud's conjecture for exponential weights *Constr. Approx.* **4** 65–83

[21] Lubinsky D S and Sharif A 1983 On the largest zeros of orthogonal polynomials for certain weights *Math. Comp.* **41** 199–202

[22] Maté A, Nevai P and Totik V 1986 Asymptotics for the zeros of orthogonal polynomials associated with infinite intervals *J. London Math. Soc.* (2) **33** 303–10

[23] Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)

[24] Mhaskar H N and Saff E B 1984 Extremal problems for polynomials with exponential weight *Trans. Am. Soc.* **285** 203–34

[25] Moak D 1981 The q-analogue of the Laguerre polynomials *J. Math. Anal. Appl.* **81** 20–47

[26] Odlyzko A M 1987 On the distribution function of spacings between zeros of the zeta function *Math. Comp.* **48** 273–308

[27] Rakhmanov E A 1984 On asymptotic properties of polynomials orthogonal on the real axis *Math. USSR Sbornik* **47** 155–93

[28] Shohat J and Tamarkin J D 1950 *The Problem of Moments* revised edn (Providence, RI: American Mathematical Society)

[29] Szegő G 1975 *Orthogonal Polynomials* 4th edn (Providence, RI: American Mathematical Society)

[30] Van Assche W 1987 *Asymptotics for Orthogonal Polynomials (Lecture Notes in Mathematics 1265)* (Berlin: Springer)

[31] Wilson J A 1980 Some hypergeometric orthogonal polynomials *SIAM J. Math. Anal.* **11** 690–701